MAT 1348 3X – Test # 2 – Spring/Summer 2016

Full Name:____

Student Number:__________________________________

Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature:__________________________________

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Question	Possible Points	Points Obtained
# 1	5	
# 2	5	
# 3	5	
# 4	5	
# 5	5	
Total	25	

Instructions:

- Print your name and student number on the first two pages.
- Verify that your copy of the test has all of its 9 pages.
- You must answer all questions. There are 5 questions for a total of 25 points.
- Write the solutions to the questions in the space provided. You may use the back of the pages if necessary.
- This a closed book test, no course notes are permitted.
- Calculators are permitted.

SHOW ALL YOUR WORK

1. (**5 pts**) Give an **indirect proof** for the following theorem:

   If $n^2 + n$ is an odd integer, then n is an odd integer.

   Solution:

   - Suppose n is an even integer.
   - Hence, by definition there exists some integer k such that $n = 2k$.
   - Hence,

   $$\begin{aligned} n^2 + n &= (2k)^2 + (2k) \\ &= 4k^2 + 2k \\ &= 2(2k^2 + k) \end{aligned}$$

   - Since $n^2 + n = 2(2k^2 + k)$, where $2k^2 + k$ is some integer, we have shown that $n^2 + n$ is even.
   - Thus, by a proof by contraposition, we have proven the theorem.

2. (**5 pts**) Give a **proof by cases** for the following theorem:

   $|x+2| + |x-4| \geq 6$ for all real numbers x.

   Solution:

   - There are four cases to consider in this scenario.
     (a) $x+2 \geq 0$ and $x-4 \geq 0$, this implies that $x \geq -2$ and $x \geq 4$. Hence, this case requires that $x \geq 4$.
     (b) $x+2 \geq 0$ and $x-4 < 0$, this implies that $x \geq -2$ and $x < 4$. Hence, this case requires that $-2 \leq x < 4$.
     (c) $x+2 < 0$ and $x-4 \geq 0$, this implies that $x < -2$ and $x \geq 4$. Hence, this case is impossible to satisfy.
     (d) $x+2 < 0$ and $x-4 < 0$, this implies that $x < -2$ and $x < 4$. Hence, this case requires that $x < -2$.
   - For each of the possible cases above we show that the inequality holds.
     (a) Consider $x \geq 4$. LHS:$|x+2|+|x-4| = (x+2)+(x-4) = 2x-2 \geq 2(4)-2 = 6$:RHS
     (b) Consider $-2 \leq x < 4$. LHS:$|x+2|+|x-4| = (x+2)-(x-4) = 6$:RHS
     (c) Impossible case to satisfy.
     (d) Consider $x < -2$, in which case $-2x > 4$. LHS:$|x+2|+|x-4| = -(x+2)-(x-4) = -2x+6 > 4+6 > 6$:RHS
   - Since the theorem holds for each case, and each case covers all the possible values of x, we have proven the theorem by cases.

3. (**5 pts**) Use the principle of mathematical induction to prove that $2^{2n} - 1$ is divisible by 3 for all integers $n \geq 0$.

Solution:

- Let $P(n)$ be the statement: $2^{2n} - 1$ is divisible by 3.

  Basis Step:

  - Since we wish to prove $P(n)$ for all integers $n \geq 0$, we show that $P(0)$ is true.
  - Since $2^{2(0)} - 1 = 1 - 1 = 0 = 3 \cdot 0$, we have shown that $P(0)$ is true.
  - This completes the basis step.

  Inductive Step:

  - Suppose that for some arbitrary integer k, $k \geq 0$, that $P(k)$ is true. I.e. $2^{2k} - 1$ is divisible by 3.
  - Thus if $n = k + 1$ we have that

$$\begin{aligned} 2^{2(k+1)} - 1 &= 2^{2k+2} - 1 \\ &= 4 \cdot 2^{2k} - 1 \\ &= 3 \cdot 2^{2k} + (2^{2k} - 1) \end{aligned}$$

  - Since the first term in the sum is a multiple of 3, the second term is divisible by 3 by the I.H., and that the sum of integers divisible by 3 is divisible by 3, we have shown $P(k+1)$ is true.
  - This completes the Inductive Step.

- By the principle of mathematical induction we have proved the theorem.

4. (**5 pts**) Let $A = \{1, \{1\}, \{1,2\}, 3\}$. For each of the following statements, indicate if it is true or false. (**You do not have to justify your answer for this question**).

   (a) $\{1, \{1\}\} \subseteq A$

   (b) $\{2, \{1,2\}\} \subseteq A$

   (c) $\{1,2\} \in P(A)$ where $P(A)$ is the powerset of A.

   (d) The cardinality of the powerset of A is 16.

   (e) $\{1,2\} \in A$.

   Solution:

   (a) T

   (b) F

   (c) F

   (d) T

   (e) T

5. (**5 pts**) Let U be the universal set and A, B, and C be three subsets of U. Use the properties of set operations to show that

$$C - (\overline{A} \cap B) = (C \cap A) \cup (C - B).$$

Solution:

(a) Using set-builder notation:

$$\begin{aligned}
& C - (\overline{A} \cap B) \\
= \; & \{x \in U \mid x \in C - (\overline{A} \cap B)\} \\
= \; & \{x \in U \mid x \in C \wedge x \notin (\overline{A} \cap B)\} \\
= \; & \{x \in U \mid x \in C \wedge \neg(x \in (\overline{A} \cap B))\} \\
= \; & \{x \in U \mid x \in C \wedge \neg(x \in (\overline{A} \wedge x \in B))\} \\
= \; & \{x \in U \mid x \in C \wedge (x \notin \overline{A} \vee x \notin B)\} \\
= \; & \{x \in U \mid (x \in C \wedge x \notin \overline{A}) \vee (x \in C \wedge x \notin B)\} \\
= \; & \{x \in U \mid (x \in C \wedge x \in A) \vee (x \in C \wedge x \notin B)\} \\
= \; & \{x \in U \mid x \in (C \cap A) \vee x \in (C - B)\} \\
= \; & \{x \in U \mid x \in ((C \cap A) \cup (C - B))\} \\
= \; & (C \cap A) \cup (C - B)
\end{aligned}$$

(b) Using algebraic manipulation:

$$\begin{aligned}
C - (\overline{A} \cap B) &= C \cap \overline{(\overline{A} \cap B)} \text{ (by definition of the difference operation)} \\
&= C \cap (\overline{(\overline{A})} \cup \overline{B}) \text{ (by De Morgan's laws)} \\
&= C \cap (A \cup \overline{B}) \text{ (by complementation law)} \\
&= (C \cap A) \cup (C \cap \overline{B}) \text{ (by distributive laws)} \\
&= (C \cap A) \cup (C - B) \text{ (by definition of the difference operation)}
\end{aligned}$$

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